LECTURE # 14. 21/11/2020.

THEOREM: Let G be a group and S be a G-set. For all $a \in S$, $[G : G_a] = |[a]|$.

Proof: Let $a \in S$. Let $\mathcal{L}$ be the set of all left cosets of G_a in G.

Now we have $[a] = \{b \in S : a \sim b\} = \{b \in S : ga = b, g \in G\}$

$\therefore$ $[a] = \{ga : \text{for some } g \in G\}$.

Let us define a function $f : \mathcal{L} \to [a]$ by

$$f(gG_a) = ga, \ \forall\, gG_a \in \mathcal{L}.$$

Let $g_1, g_2 \in G$. Then $g_1 G_a = g_2 G_a$

$\Leftrightarrow g_2^{-1} g_1 \in G_a \Leftrightarrow (g_2^{-1} g_1)a = a$

$\Leftrightarrow g_2(g_2^{-1}(g_1 a)) = g_2 a$

$\Leftrightarrow (g_2 g_2^{-1})(g_1 a) = g_2 a \Leftrightarrow g_1 a = g_2 a$

$\Leftrightarrow f(g_1 G_a) = f(g_2 G_a)$.

$\therefore$ f is well-defined and one-one function.

Let $b \in [a]$. Then $\exists\, g \in G$ s.t. $ga = b$.

$\therefore$ $f(gG_a) = ga = b \Rightarrow b$ has a pre-image gG_a in $\mathcal{L}$ under f.

$\Rightarrow f$ is onto.

$\therefore$ $f : \mathcal{L} \to [a]$ is a bijection.

$\therefore$ $|\mathcal{L}| = [G : G_a] = |[a]|$.

In particular,

If G be a finite group acting on a non-empty finite set S, and $a \in S$, then

$$|G| = |O_G(a)| \cdot |G_a|$$

This is the Orbit-Stabilizer theorem.

In the above theorem, we have shown that $[G : G_a] = |[a]|$, where $[a] \equiv O_G(a)$.

$\therefore$ $[G : G_a] = |O_G(a)|$.

If G be finite, and S be also finite, then

$[G : G_a] = \dfrac{|G|}{|G_a|}$. Hence $\dfrac{|G|}{|G_a|} = |O_G(a)|$

$\Rightarrow |G| = |O_G(a)| \cdot |G_a|$

Orbit - Stabilizer Theorem :-

Let G be a finite group acting on a non-empty finite set S, and $a \in S$. Then

$$|G| = |orb_G(a)| \cdot |Stab_G(a)| \quad \text{OR.} \quad |G| = |O_G(a)| \cdot |G_a|$$

PROOF: Let $g_1, g_2 \in G$ and $a \in S$. Then

$$g_1 a = g_2 a \Leftrightarrow g_2^{-1} g_1 a = a \Leftrightarrow g_2^{-1} g_1 \in G_a.$$
$$\Leftrightarrow g_1 G_a = g_2 G_a$$

It follows that $\longrightarrow$

The number of distinct elements in $O_G(a)$ is equal to the " " " " left cosets gG_a for $g \in G$, which is equal to $|G|/|G_a| \; (= [G : G_a])$.

$$\therefore \; |O_G(a)| = \frac{|G|}{|G_a|} = [G : G_a] \,. \quad \text{(proved)}$$

Burnside Lemma :

Let G be a finite group acting on a non-empty finite set S. Then the number of orbits of G in S is $\frac{1}{|G|} \sum_{g \in G} |S^g|$; where $|S^g| \equiv$ the number of elements of S fixed by g.

Proof: Let $S = O_G(a_1) \cup O_G(a_2) \cdots \cup O_G(a_K)$, where $\{O_G(a_1), O_G(a_2), \cdots, O_G(a_K)\}$ is the set of all distinct orbits of G on S. [Since S can be partitioned as the union of orbits].

$$\therefore \; \sum_{g \in G} |S^g| = \sum_{a \in O_G(a_1)} |G_a| + \sum_{a \in O_G(a_2)} |G_a| + \cdots + \sum_{a \in O_G(a_K)} |G_a| \,.$$

Let us suppose a, b are in the same orbit.

Then $O_G(a) = O_G(b)$ and $[G : G_a] = |O_G(a)| = |O_G(b)| = [G : G_b]$

$$\Rightarrow \frac{|G|}{|G_a|} = \frac{|G|}{|G_b|} \Rightarrow |G_a| = |G_b|$$

Thus
$$\sum_{g \in G} |S^g| = |O_G(a_1)|\,|G_{a_1}| + |O_G(a_2)|\,|G_{a_2}| + \cdots + |O_G(a_K)| \cdot |G_{a_K}| \,.$$
$$= \sum_{i=1}^{K} |O_G(a_i)|\,|G_{a_i}| = \sum_{i=1}^{K} |G| = K|G|$$

$$\therefore \; K = \frac{1}{|G|} \sum_{g \in G} |S^g| \,. \quad \boxed{K \text{ is the no. of distinct orbits}}$$

Examples of Orbits, Stabilizers.

① Let G be the permutation group defined by $G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$ and $S = \{1, 2, 3, 4, 5\}$.

Then S is a G-set.

$(1) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$, $\quad (123)(45) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$$

$(132)(45) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$.

To find Orbits:— $O_G(x) = \{g \cdot x \mid g \in G\}$
$= \{g(x) : g \in G\}$.

$\therefore \quad O_G(1) = \{g \cdot 1 / g \in G\}$, $1 \in S$,
$= \{g(1)\}$, where $g : S \to S$ is a bijection.

i.e., $O_G(1) = \{1, 2, 3\}$

Similarly, $O_G(2) = \{1, 2, 3\}$, $O_G(3) = \{1, 2, 3\}$.
$O_G(4) = \{4, 5\}$, $O_G(5) = \{4, 5\}$.

i.e., $O_G(1) = O_G(2) = O_G(3) = \{1, 2, 3\}$
and $O_G(4) = O_G(5) = \{4, 5\}$.

To find Stabilizers:— $G_x = \{g \in G : g \cdot x = x\}$, $x \in S$.

$G_1 = \{g \in G : g \cdot 1 = 1\} = \{g(1) = 1\}$.
$= \{(1), (4\ 5)\}$

$G_2 = \{g(2) = 2\} = \{(1), (45)\}$

$G_3 = \{g(3) = 3\} = \{(1), (45)\}$

$G_4 = \{g(4) = 4\} = \{(1), (123), (132)\}$.

$G_5 = \{g(5) = 5\} = \{(1), (123), (132)\}$

$G_1 = G_2 = G_3$;
$G_4 = G_5$.

To find the **fixed point sets** of S under the action of G:

$S^G = \{x \in S : g \cdot x = x, \forall g \in G\}$.

$S^{(1)} = \{1, 2, 3, 4, 5\} = S$, $S^{(123)} = \{4, 5\} = S^{(132)}$;
$S^{(45)} = \{1, 2, 3\}$; $S^{(123)(45)} = \phi = S^{(132)(45)}$.

HOME WORK — DO IT.

(2) Let $S = \{1,2,3,4,5,6\}$ and suppose that G is the permutation group given by the permutations

$$\{(1), (12)(3456), (35)(46), (12)(3654)\}.$$

Then S is a G-set.

Find (i) the orbits of S under G.

(ii) the stabilizer subgroups of G

(iii) the fixed point sets of S under the action of G.